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LIMIT THEOREMS FOR THE RANDOM WALK DESCRIBED BY THE AUTOREGRESSION PROCESS OF ORDER ONE

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Abstract. In the paper is proved central limit theorem for the value of random walk in moment of the first passage time beyond the level by a process described by a nonlinear function of autoregression process of order one ($AR(1)$).

Keywords: autoregression process of order one ($AR(1)$), random walk, first passage time, central limit theorem.

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1. Introduction

Let $\xi_n,\ n\geq 1$ be a sequence of independent and identically distributed random variables determined on some probability space $(\Omega, \mathcal{F}, P)$. As is known ([1],[5],[12],[13]) the autoregression scheme or autoregression process of order one $AR(1)$ is determined by a recurrent relation of the form

$$X_n = \beta_0 X_{n-1} + \xi_n, ,\ n\geq 1$$

where β_0 is some fixed number, and the initial value of the process X_0 is independent of innovation $\{\xi_n\}$ (see [1], [3], [5], [7]).

Autoregression schemes are widely used in applied issues of theory of random processes ([1],[5],[13]). Recently, a great attention is paid to studying boundary crossing problems for Markov's random walks described by autregression process of order one ([3]-[11])

Assume

$$T_n = \sum_{k=1}^{n} X_k X_{k-1}\ ,\ S_n = \sum_{k=1}^{n} X_{k-1}^2 \text{ and } Z_n = \frac{T_n^2}{2S_n}\ .$$

Let us consider the family of the first passage time

$$\tau_a = \inf\{n\geq 1 : Z_n \geq a\} \tag{1}$$

by the process $Z_n,\ n\geq 1$ of the level $a\geq 0$.

Note that the family of stopping moments τ_a of the form (1) was studied in the paper [3], where limit distribution of the overshoot $Z_{\tau_a} - a$ as $a \to \infty$ was found.

Similar families of the first passage time of the level by the processes X_n, T_n and S_n were considered in the papers [5]-[11], where several important properties of the family of the stopping moments of type τ_a in (1) were studied.

In the present paper we prove a central limit theorem for the processes Z_n as $n \longrightarrow \infty$ and Z_{τ_a} as $a \to \infty$.

Note that such theorems are widely used in theory of random walks with random indices, also in renewal theory and in statistical sequential analysis ([2]-[11], [14]).

2. Formulation and proof of the main result

At first we enlist the known results on axisymmetric behavior of the processes T_n, S_n and $Z_n, n \geq 1$.

In the paper [3] (see also [12]) it is shown that for $|\beta_o| < 1$

$$\frac{T_n}{n} \xrightarrow{a.s} \frac{\beta_0}{1-\beta_0^2} = \lambda_1, \; n \to \infty \tag{2}$$

$$\frac{S_n}{n} \xrightarrow{a.s} \frac{1}{1-\beta_0^2} = \lambda_2, \; n \to \infty \tag{3}$$

and

$$\frac{Z_n}{n} \xrightarrow{a.s} \frac{\beta_0^2}{2(1-\beta_0^2)} = \lambda_3, \; n \to \infty \tag{4}$$

In [12] it was proved that under conditions $|\beta_o| < 1$ and $EX_0^2 < \infty$ it holds the central limit theorem for the process $\beta_n = \dfrac{T_n}{S_n}, \, n \geq 1$:

$$\lim_{n\to\infty} P\left(\sqrt{n}(\beta_n - \beta_0) \leq x\right) = \Phi\left(x\sqrt{\lambda_2}\right), \tag{5}$$

uniformly in $x \in R$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \; x \in R.$$

It is clear that

$$Z_n = ng\left(\frac{T_n}{n}, \frac{S_n}{n}\right), \tag{6}$$

where $g(x,y)=\dfrac{x^2}{2y}$.

Taylor's second order expansion of the function $g(x,y)$ at the point (λ_1,λ_2) gives

$$g(x,y)=\lambda_3+\beta_0(x-\lambda_1)-\frac{1}{2}\beta_0^2(y-\lambda_2)+ \\ +\frac{1}{2d_2}(x-\lambda_1)^2-\frac{d_1}{(d_2)^2}(x-\lambda_1)(y-\lambda_2)+\frac{1}{2}\frac{(d_1)^2}{(d_2)^3}(y-\lambda_2)^2, \tag{7}$$

where d_1 is an intermediate point between x and λ_1 , while d_2 is an intermediate point between y and λ_2.

From (6) and (7) we get

$$Z_n=ng\left(\frac{T_n}{n},\frac{S_n}{n}\right)=n\lambda_3+n\beta_0\left(\frac{T_n}{n}-\lambda_1\right)-\frac{n}{2}\beta_0^2\left(\frac{S_n}{n}-\lambda_2\right)+\varepsilon_n \tag{8}$$

where

$$\frac{\varepsilon_n}{n}=\frac{1}{2\lambda_{2n}}\left(\frac{T_n}{n}-\lambda_1\right)^2-\frac{\lambda_{1n}}{(\lambda_{2n})^2}\left(\frac{T_n}{n}-\lambda_1\right)\left(\frac{S_n}{n}-\lambda_2\right)+\frac{1}{2}\frac{(\lambda_{1n})^2}{(\lambda_{2n})^3}\left(\frac{S_n}{n}-\lambda_2\right)^2 \tag{9}$$

where λ_{1n} is an intermediate point between $\dfrac{T_n}{n}$ and λ_1, while λ_{2n} is an intermediate point between $\dfrac{S_n}{n}$ and λ_2.

We have

$$n\lambda_3-n\beta_0\lambda_1+\frac{n\beta_0^2\lambda_2}{2}=\frac{n\beta_0^2}{2(1-\beta_0^2)}-\frac{n\beta_0^2}{(1-\beta_0^2)}+\frac{n\beta_0^2}{2(1-\beta_0^2)}=0.$$

Then from (8) we get

$$Z_n=\beta_0T_n-\frac{\beta_0^2}{2}S_n+\varepsilon_n. \tag{10}$$

To formulate the main results, we denote $Z_n^*=\dfrac{Z_n-n\lambda_3}{\sqrt{n}}$,

$$Z_{\tau_a}^*=\frac{Z_{\tau_a}-\tau_a\lambda_3}{\sqrt{\tau_a}} \text{ and } Z_a^*=\frac{Z_{\tau_a}-\tau_a\lambda_3}{\sqrt{N_a}},$$

where $N_a=\dfrac{a}{\lambda_3}$.

It holds

Theorem 1. Let $E\xi_1 = 0,\ D\xi_1 = 1,\ 0 < |\beta_0| < 1$ and $EX_0^2 < \infty$. Then

$$\lim_{n\to\infty} P(Z_n^* \le x) = \Phi(cx),\ \ x \in R$$

where $c = \dfrac{1}{|\lambda_1|\sqrt{\lambda_2}}$.

Theorem 2. Let the conditions of theorem 1 be fulfilled. Then

$$\lim_{n\to\infty} P(Z_{\tau_a}^* \le x) = \lim_{a\to\infty} P(Z_a^* \le x) = \Phi(cx).$$

For proving these theorems we need the following statements formulated as lemmas.

Lemma 1. Let $\eta_n, n \ge 2$ be a sequence of random variables such that $\eta_n \xrightarrow{a.s.} 1$ as $n \to \infty$. Then for any sequence of random variables $Y_n, n \ge 1$,

$$P(Y_n \le x) - P(Y_n \eta_n \le x) \to 0,\ n \to \infty \ \text{ for } x \in \bigcap_{n\ge1}^{\infty} C_n$$

where C_n denote the set of points of continuity the distribution functions $F_n(x) = P(Y_n \le x)$.

This lemma was proved in the paper [11].

Lemma 2. For $0 < |\beta_0| < 1$ it holds

1) $P(\tau_a < \infty) = 1$ for all $a \ge 0$;
2) $\tau_a \xrightarrow{a.s.} \infty$ as $a \to \infty$;
3) $\dfrac{\tau_a}{a} \xrightarrow{a.s.} \dfrac{1}{\lambda_3},\ a \to \infty$.

Proof. From (4) yields

$$P\left(\sup_n Z_n = \infty\right) = 1.$$

Therefore we have

$$P(\tau_a < \infty) = P\left(\sup_n Z_n \ge a\right) = 1$$

for all $a \ge 0$.

For proving statement 2) we note that the variable τ_a as a function of parameter a increases. Consequently, there exists the limit

$$\tau_\infty = \lim_{a\to\infty} \tau_a \le \infty \ \text{ for all } \omega \in \Omega.$$

Prove that $P(\tau_\infty = \infty) = 1$. Indeed, for each $n \ge 1$

$$P(\tau_\infty > n) = \lim_{a\to\infty} P(\tau_a > n) = \lim_{a\to\infty} P\left(\sup_{1\le k\le n} Z_n < a\right) = 1$$

On the other hand, $\{\tau_\infty > n+1\} \subseteq \{\tau_\infty > n\}$, and

$$\{\tau_\infty = \infty\} = \bigcap_{n=1}^{\infty} \{\tau_\infty > n\}.$$

Therefore, by the axiom on continuity of probability measure we have $P(\tau_\infty = \infty) = 1$.

Prove statement 3). From definition of variable τ_a it follows that

$$\frac{Z_{\tau_a - 1}}{\tau_a} < \frac{a}{\tau_a} \le \frac{Z_{\tau_a}}{\tau_a}. \qquad (11)$$

By statement 2), 4) and theorem 1.2 of the work [2] we have

$$\frac{Z_{\tau_a}}{\tau_a} \xrightarrow{a.s.} \lambda_3 \text{ as } a \to \infty. \qquad (12)$$

Then statement 3) of the proved lemma follows from (11) and (12).

Lemma 3. Let the sequence $Y_n, n \ge 1$ converge in distribution to the random variable Y and be uniformly continuous in probability, i.e. the following relation be satisfied:

$$\lim_{\delta\to 0} \sup_{n\ge 1} P\left\{\max_{1\le k\le n\delta} |Y_{n+k} - Y_n| \ge \varepsilon\right\} = 0 \qquad (13)$$

for any $\varepsilon > 0$.

Let $N(t),\ t \ge 0$ be the family of non-negative integer random variables such that

$$\frac{N(t)}{t} \xrightarrow{P} c \text{ as } t \to \infty,$$

where $c > 0$ is some constant.

Then the families $Y_{N(t)}$ and $Y_{[ct]}$ of random variables converge in distribution to the random variable Y as $t \to \infty$.

This lemma is one of the variants of the well known theorem of Anscombe (see e.g. [2],[14])

Lemma 4. The following statements are valid:

1) If the sequence of random variables converge almost surely to finite limit, then they are uniformly continuous in probability, i.e. (13) is satisfied.

2) If the sequence of random variables X_n and $Y_n, n \ge 1$ are uniformly continuous in probability, then the sum $X_n + Y_n, n \ge 1$ is also uniformly continuous in

probability. In addition if the sequences X_n and $Y_n, n \geq 1$ are stochastically bounded, then the product $X_n Y_n, n \geq 1$ is uniformly continuous in probability.

The proof of this lemma is given in [14].

Lemma 5. If the conditions of theorem 1 are satisfied, the sequence

$$Z_n^* = \sqrt{n}\left(\frac{Z_n}{n} - \lambda_3\right),\ n \geq 1$$

is uniformly continuous in probability.

Proof. From (10), taking into account $\lambda_3 = \beta_0 \lambda_1 - \frac{\beta_0^2}{2}\lambda_2$ we have

$$\sqrt{n}\left(\frac{Z_n}{n} - \lambda_3\right) = \beta_0 \sqrt{n}\left(\frac{T_n}{n} - \lambda_1\right) - \frac{\beta_0^2}{2}\sqrt{n}\left(\frac{S}{n} - \lambda_2\right) + \frac{\varepsilon_n}{\sqrt{n}}. \qquad (14)$$

As is shown in the paper [3], the sequences

$$T_n^* = \sqrt{n}\left(\frac{T_n}{n} - \lambda_1\right) \text{ and } S_n^* = \sqrt{n}\left(\frac{S_n}{n} - \lambda_2\right),\ n \geq 1$$

are stochastically bounded and uniformly continuous in probability.

From (9) and (14) we get

$$Z_n^* = \beta_0 T_n^* - \frac{\beta_0^2}{2} S_n^* + \frac{1}{\sqrt{n}}\left[\frac{1}{2\lambda_{1n}}\left(T_n^*\right)^2 - \frac{\lambda_{1n}}{(\lambda_{2n})^2} T_n^* S_n^* + \frac{1}{2}\frac{(\lambda_{1n})^2}{(\lambda_{2n})^3}\left(S_n^*\right)^2\right].$$

By (2) and (3) we have

$$\lambda_{1n} \xrightarrow{a.s.} \lambda_1 \text{ and } \lambda_{2n} \xrightarrow{a.s.} \lambda_2 \text{ as } n \to \infty.$$

Then the statement of lemma 5 follows from lemma 4.

Now prove the main results.

Proof of theorem 1. At first we consider the case $0 < \beta_0 < 1$

Denote

$$\eta_n = \frac{Z_n}{\lambda_3 n}.$$

It is clear that (4) yields

$$\eta_n \xrightarrow{a.s.} 1 \text{ as } n \to \infty. \qquad (15)$$

From (2) and (3) it follows that

$$\beta_n = \frac{T_n}{S_n} \xrightarrow{a.s.} \beta_0 \text{ as } n \to \infty. \qquad (16)$$

From limit relation (5) it follows that

$$P(\beta_n \leq x) - \Phi\left(\frac{\sqrt{n}(x - \beta_0)}{\sqrt{\lambda_2}}\right) \to 0 \qquad (17)$$

uniformly in $x \in R$ as $n \to \infty$.

Applying the lemma 1 for the sequence β_n we have

$$P(\beta_n \le x) - P(\eta_n \beta_n \le x) \to 0 \text{ as } n \to \infty, \tag{18}$$

where $\eta_n = \dfrac{Z_n}{\lambda_3 n}$.

Taking into account (15) and (16), from (17) and (18) we find

$$P\left(\frac{Z_n}{n} \le \frac{x\lambda_1}{2} \right) - \Phi\left(\frac{\sqrt{n}(x - \beta_0)}{\sqrt{\lambda_2}} \right) \to o \tag{19}$$

uniformly in $x \in R$ as $n \to \infty$.

From (19) instead of x we assume $x\sqrt{\dfrac{\lambda_2}{n}} + \beta_0$ and have

$$P\left(\sqrt{n}\left(\frac{Z_n}{n} - \frac{\beta_0 \lambda_1}{2} \right) \le \lambda_1 \sqrt{\lambda_2}\, x \right) - \Phi(x) \to 0$$

uniformly in $x \in R$ as $n \to \infty$.

Hence it follows that

$$P(Z_n^* \le x) - \Phi\left(\frac{x}{\lambda_1 \sqrt{\lambda_2}} \right) \to 0,\ n \to \infty.$$

This completes the proof of theorem 1 for the case $0 < \beta_0 < 1$.

Proof of theorem 1 in the case $-1 < \beta_0 < 0$ is carried out similarly and we use the equality $\Phi(-x) + \Phi(x) = 1$ for any $x \in R$.

For proving theorem 2 it suffices to note that by lemma 2, 3 and 5, its statement follows from theorem 1.

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Bir tərtibli avtoreqression proseslə təsvir olunan təsadüfi dolaşma üçün limit teoremləri

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XÜLASƏ

İşdə bir tərtibli avtoreqression $AR(1)$ prosesi ilə təsvir olunan təsadüfi dolaşmanın səviyyəni kəsmə anındakı qiymıəti üçün mərkəzi limit teoremi isbat edilir.

Açar sözlər: bir tərtibli avtoreqression proses $AR(1)$, təsadüfi dolaşma, birinci dəfə kəsmə anı, mərkəzi limit teoremi.

Предельные теоремы для случайного блуждания, описываемого процессом авторегрессии первого порядка

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РЕЗЮМЕ

В работе доказывается центральная предельная теорема для значения в момент пересечения уровня случайным блужданием, описываемом процессом авторегресси первого порядка $AR(1)$.

Ключевые слова: авторегрессионный процесс первого порядка, случайного блуждание, момент первого выхода, центральная предельная теорема.